

DOI 10.26886/2520-7474.3(53)2022.2

UDC:51-37

**PROPOSED IS A COUNTEREXAMPLE FOR THE DIAGONAL METHOD
OF GEORG CANTOR**

Mykola Stukach

<https://orchid.org/0000-0002-2001-6937>

e-mail: mykola.stukach@npp.nau.edu.ua

National Aviation University, Ukraine, Kyiv

The diagonal method proposed by Georg Cantor is a proof that unenumerable sets exist. For example, the set of all subsets of the set of all natural numbers; or the set of all binary sequences; or the set of all decimals fractions belong in gto the interval [0; 1) are allegedly not countable. This raises doubts about the axiom of choice.

For the set of all binary sequences, it is proposed to call "native" the enumeration defined by the rule: "If the reinary sequence $(a_0, a_1, a_2, a_3, \dots)$, all which elements belong to set $\{0,1\}$, then its exact position in the "native enumeration" calculates by the formula $a_0 \times 2^0 + a_1 \times 2^1 + a_2 \times 2^2 + a_3 \times 2^3 + \dots$, where \times —the multiplication sign (all arithmetic operations are performed in one of number system, such as decimal, and without limitation of digit capacity)".

Proofs are given that "native enumeration" is a counterexample for George Cantor's diagonal method.

Keywords: *discrete mathematics, counter example, enumeration of set, enuberable set, unenuberable set, fabulist Aesop, Guilherme Figueredo, mathematical intuition, diagonal method of Georg Cantor, circus trick, infinite set, cardinality of set, the set of all subsets of a set, the set of all binary sequences, axiom of choice, set of all real numbers, $1+2+3+4+\dots$*

Georg Cantor's diagonal method is considered like a model of impeccable mathematical proof [1; 2]. For example, the Penrose [3], the well-known connoisseur and popularizer of science, thought so. Unfortunately the reappeared several very unpleasant paradoxes following from the theorems, proved by this method. In addition, to bypass the axiom of choice, the axiomatic system of Zermelo Fresnel's set theory was developed, which is inconvenient to use.

Most mathematician shave not found anything better than to come to terms with this. Many mathematicians use the so-called 'naive set theory', where this these paradoxes are simply ignored. And only a few mathematicions disagree with Georg Cantor and continue to fight, mainly by looking for a counterexample.

When I heard the presentation of the diagonal method by Georg Cantor applied to the set of all decimal fractions belonging to interval [0; 1) at a lecture while studying at the Kiev Polytechnic Institute, I intuitively perceived it as a fake, as an attempt to fool everyone. (May Georg Cantor and the Gods forgive me, but that was my first feeling!) It was not so easy to formulate objections – it took me decades!

Cantor himself only by a part of his brain believed in his method, while by the other part he deeply doubted it. The proof is that he began to periodically be treated for bouts of insanity in the clinic of the same university where he taught, and even at the time of his death he was treated in that clinic.

Definition of “native enumeration”

I propose that 'a native for some set of elements placed in it means the rule: 'If an element of the set is a binary sequense $(a_0, a_1, a_2, a_3, \dots)$, then the exact position of the element in the enumeration is calculating by the formula $a_0 \times 2^0 + a_1 \times 2^1 + a_2 \times 2^2 + a_3 \times 2^3 + \dots$,

where \times – the multiplication sign (all arithmetic operations are performed in one of number system, such as decimal, without limitation of digit capacity)'.

As you can see, the “native enumeration” in terms of construction principles resembles the positional decimal number system, invented in the 5th century in India and from the end of the 8th to the beginning of the 9th centuries (after al Khorezmi proposed algorithms for four arithmetic operations performed in a column - addition, subtraction, multiplication, division) are used by Mankind without any problems.

Our goal

We will prove that “native enumeration” is a counterexample to Georg Cantor's diagonal method. The complete proof will consist of three parts.

Part 1

Why do we need the play about ancient Greek fabulist Aesop`s life?

There exists a play ‘The Fox and the Grapes’ by a Brazilian author [4] about the famous ancient Greek fabulist Aesop`s life. I saw it as a child on TV and it made a strong impression on me.

There is an episode in the play. when Aesop`s owner philosopher Xanthos (Aesop was a slave) invited the officer Agnostos to his house and, when got drunk, vowed to drink the sea up. Otherwise Xanthos will give to Agnostos housing and property. And all this was recorded on paper, which Agnostos brought to Xanthos in the morning.

Xanthos was trapped. Only a miracle could save him. Knowing Aesop as an exceptional clever person, Xanthos asked Aesop to save him, promising to give Aesop freedom in return.

After thinking for a second, he said that Xanthos should confirm that he would drink the sea up, as promised. But Xanthos did not promise to drink the rivers that flow into the sea. Let the flowing rivers stop, and Xanthos will drink the sea up.

The parallel with our case is as follows: as soon as the inflowing rivers are diverted, the sea will be drunk up. As soon as you tell me the supposedly missing element in the enumerating set of all binary sequences, I will tell you that you are wrong and show you the position of this element in the enumerating set. This is happening because the supposedly missing element itself will indicate me its exact position as described above.

Part 2

Detective investigation: ‘How did it happen that Georg Cantor’s diagonal method deceived a lot of people, starting with me?’

(In fact, the method fooled almost everyone, but not everyone. Some persons struggled with it desperately.)

As I said, I got acquainted with the diagonal method of Georg Cantor at a lecture at the Kiev Polytechnic Institute, where this method was applied to the set of all proper decimal fractions (further denoted as the set **APDF**).

My guess is this: it was a trick. It took place according to the usual trick scheme: first, the magician diverts the attention of the audience through various passes, and then secretly uses a dishonest trick.

In the case of the diagonal method, the audience was distracted by the fact that the demonstration of the diagonal method took place on a set of infinite decimal fractions in which the digits were changed in a complicated way.

The secret use of a dishonest trick was that it has then announced that we had changed all decimal fractions in the course of the diagonal method and there appeared many fractions that were not in the original enumeration.

There is an error here (either conscious or unconscious). Georg Cantor could forget about this or not remember, that is, the trick could happen by it

self. If it did not happen by it self, one should also say the professional phrase of all magicians: "Sleight of hand, and no cheating!" ;-).

So what was the dishonest trick? It was

using of the
multiset **APDF** instead
of using the
set **APDF** and then
pumping the set
APDF with copies of
its content.

Remark: “Amultiset” differs from “ a set “in that it admits the repetition of its elements. This means that the set of non-repeating elements of both sets can coincide, and the set of all elements of a multiset can be arbitrarily larger than that of a set. By repeated application of Cantor's diagonal method, we can wind up the number of elements of the multiset to an arbitrarily large value. This diagonal method is dangerous, it is incorrect.

So, the focus under consideration is organized like this. 1) It is assumed that there is a container containing all elements of some infinite type. The diagonal method is applied to this container, changing many-many elements of the stored type. This results in appearance many-many new elements of the stored type. They were not in the container, so they need to be added.

And at that moment, the container literally explodes from the catastrophic appearance of a lack of space inside it. Although at first (before applying the diagonal Cantor`s method), all its elements, as it was announced, fit.

The audience is in amazement, the same as if a magician pulled a rabbit out of an empty hat, as usually happens at such performances.

2) Since it turned out that there is no extra space in the container, we need a new, much larger container.

The reason for this may be that we now count all elements of the multiset, and not just unique ones.

To find it out, you can use the container reweighing tactic: weigh the container before and after using the diagonal method. To weigh the container (that is, to determine the power of the set of elements in it), we have a convenient tool – “native enumeration”. This is because “native enumeration” corrects attempts to turn the set placed in it into a multiset, i.e. occurrence of repetitive elements.

To introduce the “native enumeration of the set of all proper decimal fractions”, let’s move from the “set of all decimal fractions” to the “set of all proper binary fractions” and for each fraction to add to it in the form of a label its “native number”, which we obtain from the notation of the binary fraction by remove “0.” at its beginning.

Since the content of the container both before and after applying the diagonal method successfully receives “native enumeration”, we conclude that the cardinality of the set **APDF** contained in the container does not change.

Philosophical remark:

In fact, from the point of view of philosophy, Georg Carter substituted actual infinity for potential. This is the true source of the falseness of Cantor’s diagonal method. The fact that actual and potential infinity cannot be interchanged can be seen, among other things, in the example of Ramanujan’s formula for the sum of series of natural numbers (i.e. $1+2+3+4+\dots=-1/12$). While infinity is actual (the series is not interrupted), it is equal to $-1/12$ in total. As soon as the series is interrupted, i.e. infinity manifests its potentiality, the series turns into potentially any giant positive integer.

We can not impose our views on Nature. We can only study it, discover its laws. Because we cannot take into account all the factors, all its goals. This is the philosophy of antiquity Plato. According to Plato, the basis of Nature is the world of absolutely perfect ideas that we cannot grasp with one glance. We can only discover them for ourselves, and even then in parts. Among them is the Heisenberg uncertainty, for example, which is needed by Nature at least for an electron in an atom does not fall on the nucleus. Or Einstein’s theory of relativity with its four-dimensional space (3 spatial coordinates and the imaginary coordinate of time), so that the speed

of light is constant. Or mysterious quantum entanglement. For one reason or another, Nature needs all this exactly in this form!

Part 3

Confirmation that the diagonal method of Georg Cantor has a fundamental flaw that we have previously unraveled

Using Cantor's diagonal method, we'll prove that the set of all binary sequences (let's call it the set **ABS**) is non-enumerable. First, we make the assumption that the set **ABS** is enumerable. Let it has "native enumeration", i.e. if there is a binary sequence $(a_0, a_1, a_2, a_3, \dots)$, all which elements belong to set $\{0,1\}$, then the exact position of the sequence in the "native enumeration" calculates by the formula $a_0 \times 2^0 + a_1 \times 2^1 + a_2 \times 2^2 + a_3 \times 2^3 + \dots$, where \times —the multiplication sign (all arithmetic operations are performed in one of number system, such as decimal, and without limitation of digit capacity)". The binary sequences that belong to the set **ABS** before starting the application of Cantor's diagonal method will be denoted by b_0, b_1, b_2, \dots .

Now let's apply Cantor's diagonal method, i.e. let i run through the sequence of values $=0, 1, 2, 3, \dots$. At the i -th step, we change the value of the i -th bit of the sequence b_i to the opposite, then we conclude that in addition to the old sequence b_i already stored in the set **ABS**, as a result of the described bit change, a new sequence B_i has appeared, which does not yet belong to the set **ABS**, on the basis of which we should add the sequence B_i to the set **ABS**.

And they would add (in the sense they would take into account again in the calculation of the number of elements of the set **ABS**), if they dared to ignore that the new sequence B_i has a position in our enumeration, i.e. already present in the set **ABS**. We are not allowed to add anything. The proof collapsed. It is incorrect.

Remark: I have a lot of respect for programmers who write in the Java, C++, C# and the like, as well as in the Haskell, OCaml, ML, Scheme and the like. I consider them “practicing mathematicians of such branch of mathematics as Constructivism”.

They develop a strong professional sense of what the computer can and cannot do. In addition, they are usually distinguished by their intelligence and ingenuity. They quickly translate any solution from a theoretical reasoning into a practical reasoning.

I hope for their understanding in the matter under consideration.

Conclusions

It looks like there are no non-enumerable sets. The set of all binary sequences has the largest cardinality of all possible infinite sets. The set of all subsets of the set of all natural numbers has one less cardinality (the set of all subsets itself is discarded to eliminate Bertrand Russell's paradox). The set of all natural numbers is of the same cardinality (0 is not considered a natural number). The set of all binary sequences, the set of all decimal fractions belonging to the interval $[0; 1)$, and the set of all subsets of the set of all natural numbers are enumerable. The axiom of choice is true.

To work correctly with the infinite sets invented by Cantor, you can use the appropriate program. It can be written in Java. Including to store the studied set, you can use the corresponding collection.

For example, if we were talking about creating a Java program to implement a container that stores “native enumeration”, I would choose the Array List library collection. If you are confused that Java does not support infinite sets by means of the language itself, you can use (instead of Java) some functional programming language that has built-in supports lazy

evaluation and, accordingly, with built-in support for potentially infinite sets. It could be Haskell.

(An example of a Haskell program using infinite sets of sieve of Eratosthenes:

```
primes = sieve [2..] where sieve (x:xs) = x:sieve (filter ((/=0).( `mod` x)) xs)
```

Here [2..] is an infinite set in the form of a sequence 2, 3, 4, 5, ...).

(True Java fans believe that their programming language is good enough for all applications, and if something is not supported in Java at the language level, then it is either supported as a ready-made solution at the level of the language libraries, or they will add the necessary program code themselves.)

Remark: Taking into account my experience (I immediately doubted Cantor's diagonal method, and decades later I came up with a counterexample), I must say: the counterexample for the Georg Cantor's diagonal method is proposed, but you still have to believe in it.

The last clarification is due to the fact that I really very respect the mathematical intuition of the human race.

Further actions

So, the counterexample is proposed. It remains to believe in this decision. (I admit that many persons will not believe in it.

Although, for example, I do not object to the formula $1+2+3+4+\dots=-1/12$ by the Indian genius Ramanujan after I found out that 3 physical theories being based on it had received Nobel Prizes ;-).

In general, let's recap: the counterexample for the Georg Cantor's diagonal method is proposed, but you still have to believe in it.

References:

1. Cantor`s diagonal argument. Retrieved from https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument (2022, September, 22).
2. Cantor diagonal method. Retrieved from <https://mathworld.wolfram.com/CantorDiagonalMethod.html> (2022, September, 22)
3. Roger Penrose. Retrieved from https://en.wikipedia.org/wiki/Guilherme_Figueiredo (2022, September, 22).
4. Guilherme Figueredo. Retrieved from https://en.wikipedia.org/wiki/Guilherme_Figueiredo (2022, September, 22).
5. Kapadia, Devendra (2014). TheABCDofdivergent series. Retrieved from <https://blog.wolfram.com/2014/08/06/the-abcd-of-divergent-series>. (2022, September, 22).
6. Stukach, Mykola (2022). To save mankind: proposed is an alternative to mining blockchains. - New York. TK Meganom LLC, Innovative Solutions in Modern Science, 1(53), pp. 39-47, ISSN 2414-634X, www.naukajournal.org/index.php/ISMMSD [in English]

Citation: Mykola Stukach (2022). PROPOSED IS A COUNTEREXAMPLE FOR THE DIAGONAL METHOD OF GEORG CANTOR. Frankfurt. TK Meganom LLC. Paradigm of knowledge. 3(53). doi: 10.26886/2520-7474.3(53)2022.2

Copyright Ihor Horbach ©. 2022. This is an openaccess article distributed under the terms of the Creative Commons Attribution License (CC BY). The use, distribution or reproduction in other forums is permitted, provided the original author(s) or licensor are credited and that the original publication in this journal is cited, in accordance with accepted academic practice. No use, distribution or reproduction is permitted which does not comply with these terms.